**Orthogonal vectors**

* Geometrically, **a basis** is **a set of coordinate axes**.
* In order to convert geometric constructions into **simple** algebraic aclculations
  + 1. **the length ||*x*||** of a vector = 1
  + 2. ***xTy* = *0*** for perpendicular vectors
* **the length ||***x***||** in **Rn** is the positive square root of *x****T****x*
  + **length squared**: **||*x*||2** = *x12 + x22 + … + xn2 =* ***xTx***
* **Inner product**: *x****T****x = (x,y) = x\*y = matrics multiplication =x1y1 + … + xnyn*
  + scalar product or dot product
* **Orthogonal vectors:** the inner product ***xTy*** is **zero** ⇐⇒ ***x*** and ***y*** are **orthogonal vectors**.
  + ***xTy > 0,*** angle is less than **90*; xTy < 0,*** angle is greater than **90**
  + **length squared** is the **inner product** of *x* with itself: ***xTx***
  + **zero vector** is orthogonal to every vector in ***Rn***
    - **zero vector** is the **only vector** orthogonal to itself
* If **nonzero vectors** *v1, … , vk* are **mutually orthogonal**, those **vectors** are **linearly independent**.
  + **Orthogomal vectors** in ***R2***: *v1 = (cos****𝜃****, sin****𝜃****) and v2 = (-sin****𝜃****, cos****𝜃****)*

**Orthogonal subspaces**

* two subspaces ***V*** and ***W*** of the same space ***Rn*** are orthogonal if every vector ***v*** in ***V*** is orthogonal to every vector ***w*** in ***W***: ***vTw*** = ***0*** for all ***v*** and ***w***.
  + the subspace {***0***} orthogonal to all subspaces.
  + **A line** can be orthogonal to another line, or be orthogonal to a plane.
  + **A plane cannot** be orthogonal to another plane(in three dimensions).
* **Fundamental theorem of orthogonality**
  + **the row space ⊥**  **the nullspace** (in ***Rn***): ***N****(****A****)⊥****C****(****AT****)*
  + **the column space ⊥ the left nullspace** (in ***Rm***): ***N****(****AT****)⊥****C****(****A****)*
* **Orthogonal complement of V**
  + **Definition**: Given a subspace ***V*** of ***Rn***, **the space** of **all** **vectors** orthogonal to ***V*** is called the **orthogonal complement** of ***V***, denoted ***V***⊥ =***V*** *perp.*
  + If ***W = V*⊥** then ***V = W*⊥** or ***V*⊥⊥ *= V***, dim***V*** + dim***W*** = *n*
  + **Fundamental theorem of linear Algebra, II**
    - **the nullspace** is the orthogonal complement of the **row space** in ***Rn***
    - **the left nullspace** is the orthogonal complement of the **column space** in ***Rm***
  + **Dimension formula**: **dim**(row space) + **dim**(nullspace) = **number** of columns
  + **row rank = column rank**
* ***Ax = b* is solvable**
  + 1. Directly, ***b*** is in the column space.
  + 2. Indirectly, ***b*** is perpendicular to the left nullspace: easier to verify than 1.

**Projection**

* If dimensions are right, the whole space ***Rn*** is being decomposed into two perpendicular parts
  + split **every vector** into ***x* = *v* + *w***
    - the vector ***v*** is the projection onto the subspace ***V***
    - the vector ***w*** is the projection of x onto ***W***

**what is happening inside the multiplication** *Ax( =b)*

* **the projections** of the vector space of unknown x: ***x = xr + xn***
  + ***xr*** is the projection of the unknown vector x onto the **row subspace**
    - **the nullspace component** goes to zero: ***Axn = 0***
  + ***xn*** is the projection of x onto **the nullspace**
    - **the row space component** goes to the column space: ***Axr = Ax***
* **every matrix transforms its row space onto its column space**
  + every **vector *b*** in column space comes from exactly **one** **vector *xr*** in the row space
  + from the **row space** to the **column space**, ***A*** is actually **invertible**
    - on those **r-dimensional spaces** A is invertible
      * the **invertible submatrix** holding the r nonzeros.
    - on its **nullspace** ***A*** is zero
* ***AT*** goes in the opposite direction: from ***Rm*** to ***Rn***: from ***C(A)*** to ***C(AT)***
  + the transpose in not the inverse
  + ***AT*** moves spaces correctly, but **not** **the individual vectors**
  + ***A-1*** transpose in reverse, but ***A-1*** cann’t bring back nullspace out of the **zero vector**.
    - when ***A-1*** exists, only if r=m=n: **two subspaces can transpose in both directions** (because no vectors goes into ***0***)
    - when ***A-1*** doesn’t exists, **pseudoinverse *A+*** is the best substitute.
      * it inverts ***A*** where that is possible
        + ***Axr = Ax*** ⇒ ***A+Ax*** =(***A+A)xr*** back to row space
      * but **nothing can be done** on the **left nullspace**
        + ***Axn = 0*** ⇒ ***A+0*** can not bring back to **nullspace** (***A+A)xn***

